PS1, 1, 2, 3, 4, 5

$F_X(x) = \mathbb{P}(X \leq x)$, $A_x = \{\omega \in \Omega : X(\omega) \leq x\}$

1. $\mathbb{P}(X > x) = 1 - F_X(x)$ | $A_x^c = \{\omega \ldots : X(\omega) > x\}$

2. $\mathbb{P}(X = x) = F_X(x) - F_X(x^-)$ | $\lim_{z \uparrow x} F_X(z)$

3. $\mathbb{P}(X < x) = F_X(x^-)$

$P(X = x)$ $\{\omega \in \Omega : X(\omega) = x\}$

$\{\omega \in \Omega : X(\omega) < x\}$

4. $\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)$ | $\{\omega \ldots : a < X(\omega) \leq b\} =$

$\{\omega \ldots : X(\omega) > a\} \cap \{\omega \ldots : X(\omega) \leq b\}$

Discrete RV

X is discrete if it takes countably many values $\underbrace{\{x_1, x_2, \dots\}}_{S}$

- Probability Mass Function (PMF)

$$f_X(x) = \mathbb{P}(X=x)$$
$$= F_X(x) - F_X(x^-)$$

$f_X(x) \in [0,1] \quad \forall x$

$\sum_x f_X(x) = 1$

$x \in S$, 0 otherwise

We can recover the CDF: $F_X(x) = \sum_{z: z \leq x} f_X(z)$

Continuous RVs

X is cts if its CDF can be written as

$$F_X(x) = \int_{-\infty}^{x} f_X(z)\, dz$$

where $f_X(x)$ is an integrable function. $f_X(x)$ is the probability density function (PDF)

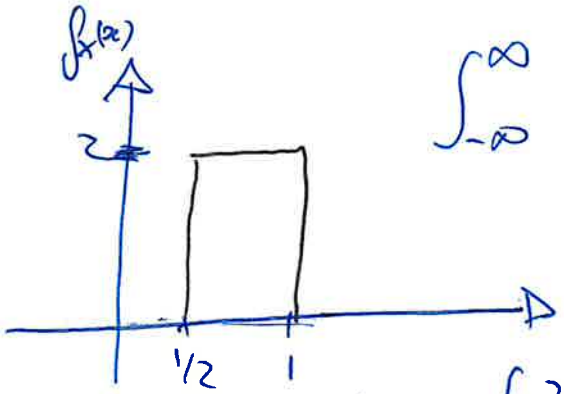
$$f_X(x) = \frac{d}{dx} F_X(x) = \lim_{h \to 0} \frac{F_X(x+h) - F_X(x)}{h}$$

$$= \lim_{h \to 0} \frac{\mathbb{P}(x < X \leq x+h)}{h}, \quad \mathbb{P}(X=x) = 0$$

PDF

$f_X(x) \geq 0$, $\cancel{f_X(x) \leq 1}$

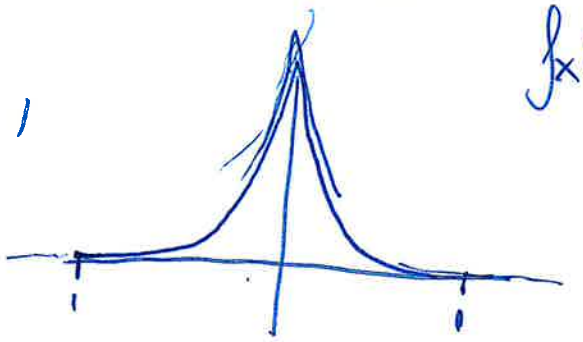
$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$



$$\int_{-\infty}^{\infty} = \int_{1/2}^{1}$$

$$f_X(x) = \begin{cases} 2 & x \in [1/2, 1] \\ 0 & \text{otherwise} \end{cases}$$

PDF is not bounded by 1



Expectation

Example: X takes n different values all with probability $\frac{1}{n}$
$\{x_1, \dots, x_n\}$

What do you expect of X?

$$\frac{1}{n} \sum_{i=1}^{n} x_i$$

X is discrete

$$\mathbb{E}[X] = \sum_{x \in S} x f_X(x),$$

X is continuous

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$\mathbb{E}[X]$ | assume $\mathbb{E}[|X|] < \infty$

Law of the Unconscious Statistician (LOTUS)

Unconscious

Suppose we have $g : \mathbb{R} \to \mathbb{R}$ that is "nice"
Given X
Then $g(X)$ is going to be a RV

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \quad \text{if continuous}$$

$$\mathbb{E}[g(X)] = \sum_{x \in S} g(x) f_X(x)$$

$$\boxed{\int a f(x)\,dx = a \int f(x)\,dx}$$

Assuming $\mathbb{E}[|g(X)|] < \infty$

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2\right]$$

$$= \mathbb{E}[X^2] - \mathbb{E}\left[2X\mathbb{E}[X]\right] + \mathbb{E}\left[\mathbb{E}[X]^2\right]$$

$$= \mathbb{E}[X^2] - 2\,\mathbb{E}[X]^2 + \mathbb{E}[X]^2$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\int_{-\infty}^{\infty} x^2\,dF_X(x) = \begin{cases} \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \\ \sum x^2 f_X(x) \end{cases}$$

k-th moment : $\int_{-\infty}^{\infty} x^k \, dF_X(x)$

k-th central moment : $\int_{-\infty}^{\infty} (x - \mathbb{E}[X])^k \, dF_X(x)$

~~Jensen~~

Jensen's Inequality

Suppose $g: \mathbb{R} \to \mathbb{R}$ is a convex function, X is a RV

Then $\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$

Proof: Since g is convex you can find m : $g(x) \geq g(x_0) + m(x - x_0)$

$x_0 = \mathbb{E}[X]$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \geq \int_{-\infty}^{\infty} \left[g(\mathbb{E}[X]) + m(x - \mathbb{E}[X]) \right] f_X(x)\, dx$$

$$= g(\mathbb{E}[X]) \underbrace{\int_{-\infty}^{\infty} f_X(x)\, dx}_{1} + m \underbrace{\int_{-\infty}^{\infty} (x - \mathbb{E}[X]) f_X(x)\, dx}_{\mathbb{E}[X] - \mathbb{E}[X] \int_{-\infty}^{\infty} f_X(x)\, dx}$$

$$= g(\mathbb{E}[X])$$

$g(x) = x^2$, then $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2 \Rightarrow \mathrm{Var}(X) \geq 0$

Moment Generating Function (MGF)

X the MGF is defined as

$$M_X(t) = \mathbb{E}\left[e^{tX}\right]$$

if it exists for ~~some~~ $t \in (-h, h)$, $h > 0$

$$\frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \mathbb{E}\left[X^k\right]$$

- MGF uniquely determines X, so if X_1 & X_2 are RV with

$$M_{X_1}(t) = M_{X_2}(t) \quad \forall t$$

~~$X_1 = X_2$~~ $\Rightarrow X_1 = X_2$

$$\frac{d^k}{d\cancel{x}t^k} M_X(t)\Big|_{t=0} = \mathbb{E}[X^k]$$

$$\mathbb{E}[e^{tX}] \quad , \quad \boxed{e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}}$$

$$\mathbb{E}[e^{tX}] = \mathbb{E}\left[\sum_{j=0}^{\infty} \frac{(tX)^j}{j!}\right] = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mathbb{E}[X^j]$$

$$\mathbb{E}[X^0] = \int_{-\infty}^{\infty} 1 \, dF_X(x) = 1$$

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = \mathbb{E}[X] + \underbrace{\sum_{j=2}^{\infty} \frac{t^{j-1}}{(j-1)!} \mathbb{E}[X^j]}_{0} \Big|_{t=0}$$

$$\frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \underbrace{\frac{k(k-1)\ldots 1}{k!}}_{1} \mathbb{E}[X^k] + \underbrace{\sum_{j=k+1}^{\infty} \frac{t^{j-k}}{(j-k)!} \mathbb{E}[X^j]}_{0} \Big|_{t=0}$$

$$= \mathbb{E}[X^k]$$