Lecture 3

$X \sim \text{Bernoulli}(p)$ if an event occurs with probability p.

PMF

$$f_X(x) = \begin{cases} p & \text{if } x = 1 \quad (\text{success}) \\ 1-p & \text{if } x = 0 \quad (\text{failure}) \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = p \cdot 1 + (1-p) \cdot 0$$

$$\mathbb{E}[X^2] = p \cdot 1^2 + (1-p) 0^2 = p$$

$$\text{Var}(X) = p - p^2 = p(1-p)$$

$$\mathbb{E}[e^{tX}] = pe^{t \cdot 1} + (1-p) e^{t \cdot 0} = \boxed{pe^{t} + (1-p)}$$

$$\Rightarrow \mathbb{E}[X^k] = p$$

$X \sim \text{Bin}(n, p)$, if X # successes in n Bernoulli trials

$X \in \{0, 1, \ldots, n\}$

$$f_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\, k!}$$

$$M_X(t) = \sum_{j=0}^{n} \binom{n}{j} e^{tj} p^j (1-p)^{n-j}$$

Binomial Expansion

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$$

$$= \sum_{j=0}^{n} \binom{n}{j} (e^t p)^j (1-p)^{n-j}$$

$$= \left(e^t p + (1-p)\right)^n$$

$$M_X'(t)\Big|_{t=0} = npe^t \left(e^t p + (1-p)\right)^{n-1}\Big|_{t=0} = np$$

$$\boxed{e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n}$$

$n \to \infty$, $np = \lambda$

$p \downarrow 0$

$$\lim_{\substack{n\to\infty \\ p\downarrow 0 \\ np=\lambda}} \left(e^t p + (1-p)\right)^n = \lim_{\substack{n\to\infty \\ \left(\substack{p\downarrow 0 \\ np=\lambda}\right)}} \left(1 + \frac{\lambda}{n}(e^t - 1)\right)^n$$

$$= \boxed{\exp\{\lambda(e^t - 1)\}}$$

Poisson Distribution

X count of occurrences of an event in some interval of time (1)

$\cancel{f_X(x) = e^{\lambda}}$

$X \sim \text{Poisson}(\lambda)$

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

$S(X) = \{0, 1, \ldots\}$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{e^t \lambda} = \exp\{\lambda(e^t - 1)\}$$

$X(t) \sim \text{Poisson}(\lambda t)$

$$M_X'(t)\Big|_{t=0} = \lambda e^t M_X(t)\Big|_{t=0} = \lambda$$

Geometric Distribution

$X \sim Geom(p)$

X is # of failures until the first success

$S(X) = \{1, 2, \ldots\}$

$$f_X(x) = \begin{cases} p(1-p)^{x-1} & x \in \{1, \ldots\} \\ 0 & \text{otherwise} \end{cases}$$

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} \left[(1-p)e^t\right]^{x-1}$$

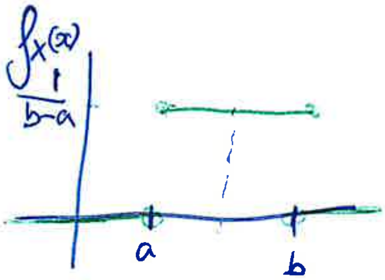
$$= pe^t \frac{1}{1-(1-p)e^t}$$

$(1-p)e^t < 1 \iff \log\left(\frac{1}{1-p}\right) > t$

$$\mathbb{E}[X] = \frac{1}{p}$$

Uniform

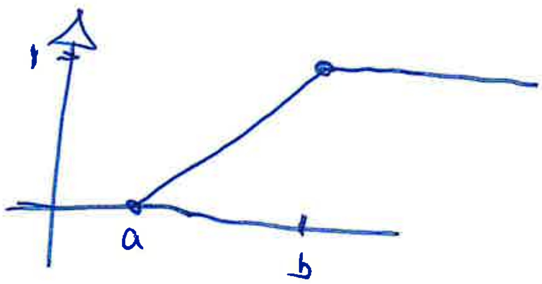
$$X \sim \text{Uniform}(a,b)$$



$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{b-a} \cdot \mathbb{1}_{\{a \le z \le b\}} \, dz = \int_a^x \frac{1}{b-a} \, dz = \frac{1}{b-a} \left[z \right]_a^x$$

$$= \frac{x-a}{b-a}$$



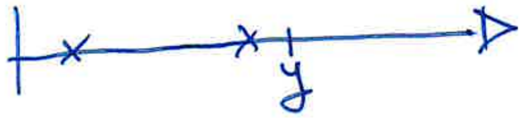
$$E[X] = \frac{b+a}{2} \quad \left(\int_a^b \frac{x}{b-a} \, dx = \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b = \frac{1}{2} \frac{b^2 - a^2}{b-a} = \frac{a+b}{2} \right)$$

Exponential

$X(t) \sim \text{Poisson}(\lambda t)$

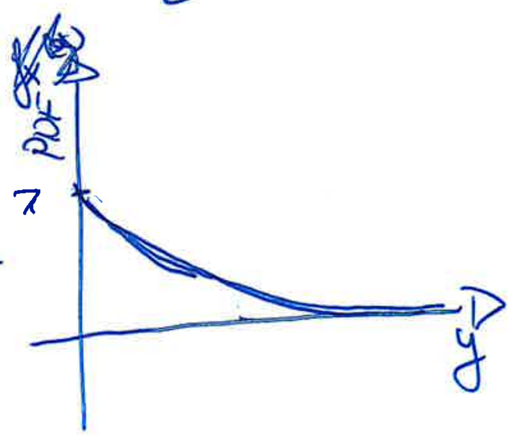
$$\mathbb{P}(X(t)=x) = \frac{e^{-\lambda t}(\lambda t)^{x}}{x!}$$

Y the time of the first arrival



$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) \\ &= 1 - \mathbb{P}(Y > y) \\ &= 1 - \mathbb{P}(X(y) = 0) \\ &= 1 - \frac{e^{-\lambda y}(\lambda t)^{0}}{0!} \\ &= 1 - e^{-\lambda y} \end{aligned}$$

$$f_x(x) = \begin{cases} \lambda e^{-\lambda y} & > 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\int_0^{\infty} x \lambda e^{-\lambda x} \, dx$$

$X \sim Exp(\lambda)$

$M_X(t)$?

$$M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \frac{\lambda}{t-\lambda} \int_0^\infty (t-\lambda) e^{(t-\lambda)x} dx$$

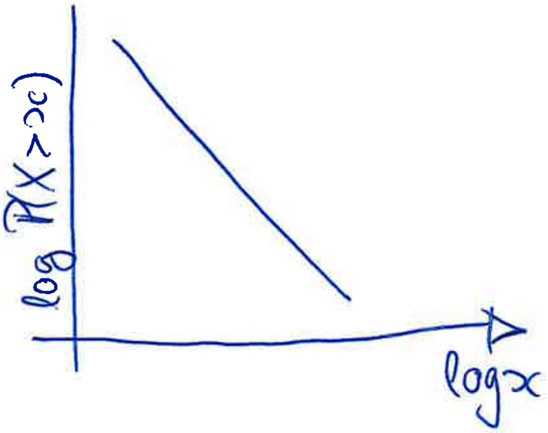
$$= \frac{\lambda}{t-\lambda} \left[e^{(t-\lambda)x} \right]_0^\infty = \frac{\lambda}{\lambda - t} \quad , \boxed{t < \lambda}$$

<u>Memory</u>

$$\mathbb{P}(X > s+t \mid X > s) = \frac{\mathbb{P}(X > s+t \cap X > s)}{\mathbb{P}(X > s)}$$

$$= \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda(s)}} = \mathbb{P}(X > t)$$

$X \sim Exp(\lambda)$, then $\mathbb{P}(X > x) = e^{-\lambda x}$



$$\log(\mathbb{P}(X>x)) = \log(K) - \alpha \log(x)$$

$$\mathbb{P}(X>x) = K x^{-\alpha}$$

$$\frac{\partial}{\partial x}\left(1-\mathbb{P}(X>x)\right) = \alpha K x^{-(\alpha+1)} \quad , \quad \alpha > 0$$

$$\int_{x_m}^{\infty} \alpha K x^{-(\alpha+1)} dx = \alpha K \left[\frac{x^{-\alpha}}{-\alpha}\right]_{x_m}^{\infty} = -K\left[0 - x_m^{-\alpha}\right] = 1$$

$$\Rightarrow K = x_m^{\alpha}$$

$$f_X(x) = \begin{cases} \dfrac{\alpha x_m^{\alpha}}{x^{\alpha+1}} & \text{for } x > x_m > 0 \\ 0 & \text{otherwise} \end{cases}$$

MGF does not exist $\forall t$ $\quad K\int_{x_m}^{\infty} e^{tx} x^{-(\alpha+1)} dx$

Gabaix (2009) "Power Laws in Economics and Finance"

ARE

Functions of Random Variables

X is a RV and g is a nice function, then $g(X) = Y$ is a RV

$X \sim$ Dist , $Y \sim ?$

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y)$$

$$\{\omega \in \Omega : g(X(\omega)) \leq y\}$$

Inverse Image:

$B \subset \mathbb{R}$

$$g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\}$$

In our case $B = (-\infty, y]$

$$F_Y(y) = \mathbb{P}\left(X(\omega) \in g^{-1}((-\infty, y])\right)$$

$$\{\omega \in \Omega : g(X(\omega)) \in (-\infty, y]\}$$

X is cts and $g(x)=x^2$, $Y=X^2$

$$\{\omega\in\Omega : X\in g^{-1}((-\infty,y])\} = \{\omega\in\Omega : \overset{X\in(0,\sqrt{y}]\cup[-\sqrt{y},0)}{\cancel{X\in g^{-1}((0,\sqrt{y}])}}\}$$

$$\mathbb{P}(X\in g^{-1}((0,y])) = \mathbb{P}(\overset{X\in(0,\sqrt{y}]\cup[-\sqrt{y},0)}{\cancel{X\in g^{-1}((0,\sqrt{y}]\cup[-\sqrt{y},0))}})$$

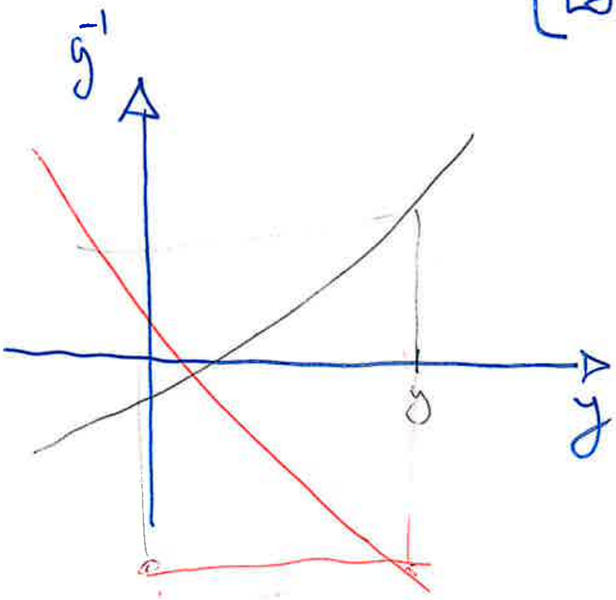
$$= \mathbb{P}(-\sqrt{y}\le X\le\sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}) = F_Y(y)$$

<u>Monotone Transformations</u>

If $g:\mathbb{R}\to\mathbb{R}$ is strictly monotonic

$$g^{-1}((-\infty,y]) = \begin{cases} (-\infty, g^{-1}(y)] & \text{if } g \text{ is increasing} \\ [g^{-1}(y),\infty) & \text{if } g \text{ is decreasing} \end{cases}$$



$Y = g(X)$

$$P(Y \leq y) = \begin{cases} P(X \leq g^{-1}(y)) & \text{if } g \text{ is increasing} \\ P(X \geq g^{-1}(y)) & \text{if } g \text{ is decreasing} \end{cases}$$

$$= \begin{cases} F_X(g^{-1}(y)) & \text{"} \quad \text{"} \\ 1 - F_X(g^{-1}(y)^-) & \text{"} \quad \text{"} \end{cases}$$

If X is continuous, we can derive the PDF

$\cancel{f_X(y) = f_X(g^{-}}$

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) & g \text{ increasing} \\ -f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) & g \text{ decreasing} \end{cases}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

Location-Scale Transform

Z is a RV, $X = \mu + \sigma Z$, $g(x) = \mu + \sigma x$

$$g^{-1}(y) = \frac{y-\mu}{\sigma}$$

$$f_X(x) = f_Z\left(\frac{x-\mu}{\sigma}\right)\left|\frac{1}{\sigma}\right|$$

Gaussian or Normal RV

$X \sim N(\mu, \sigma^2)$ $\quad f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$

$Z \sim N(0,1)$ $\quad f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\}$

If Z is $N(0,1)$, then $X = \mu + \sigma Z$ is $N(\mu, \sigma^2)$

$$\cancel{f_X(x)=} \quad \mathbb{E}\left[e^{tZ}\right] = e^{\frac{t^2}{2}}$$

$$\mathbb{E}\left[e^{tX}\right] = \mathbb{E}\left[e^{t(\mu+\sigma Z)}\right] = e^{t\mu} M_Z(t\sigma) = e^{t\mu + \frac{\sigma^2 t^2}{2}}$$