Lecture 4

CDF of X, Y (RVs)

$$F_{XY}(x,y) = \mathbb{P}(X \le x \cap Y \le y) = \mathbb{P}(X \le x, Y \le y)$$

$$\{\omega \in \Omega : X(\omega) \le x\} \cap \{\omega \in \Omega : Y(\omega) \le y\}$$

Properties

1. Limits:

$$\lim_{x,y \to \infty} F_{XY}(x,y) = 1$$

$$\lim_{x \to -\infty} F_{XY}(x,y) = 0 \quad , \forall y \in \mathbb{R}$$

vice versa

$$\lim_{x \to \infty} F_{XY}(x,y) = \lim_{x \to \infty} \mathbb{P}(X \le x \cap Y \le y) = \mathbb{P}(\Omega \cap Y \le y) = \mathbb{P}(Y \le y)$$

$$= F_Y(y) \quad (\text{marginal cdf})$$

2. F_{XY} is non-decreasing in x, y

3. F_{XY} is right-continuous in x, y

Bivariate PMF

Recall $f_X(x) = \mathbb{P}(X = x)$

$$f_{XY}(x,y) = \mathbb{P}(X = x, Y = y)$$

$$F_{XY}(x,y) = \sum_{x' \leq x} \sum_{y' \leq y} f_{XY}(x', y')$$

		Y: 1	2	3	$f_X(x)$
X	0	0.2	0.1	0.2	0.5
	1	0.3	0.1	0.1	0.5
	$f_Y(y)$	0.5	0.2	0.3	

$$f_X(x) = \sum_y f_{XY}(x,y)$$

$$\sum_x \sum_y f_{XY}(x,y) = 1$$

Bivariate PDF

X, Y are jointly continuous, the PDF is an integrable function

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy$$

1. $f_{XY}(x,y) \geq 0 \quad \forall x, y \in \mathbb{R}$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$
3. $\mathbb{P}((X,Y) \in B) = \iint_B f_{XY}(x,y)\,dx\,dy$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \quad , \quad \text{viceversa}$$

Joint Moments

X, Y are RVs

$g(X,Y)$ is also a RV if g is "nice"

$$\mathbb{E}[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y) f_{XY}(x,y) & \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) & \text{continuous} \end{cases}$$

Suppose X is discrete

$$\mathbb{E}[g(X,Y)] = \sum_x \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\,dy$$

Covariance

$$\mathrm{Cov}(X,Y) = \mathbb{E}\Big[\big(X-\mathbb{E}[X]\big)\big(Y-\mathbb{E}[Y]\big)\Big] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$$

$$\mathrm{Cov}(aX,Y) = a\,\mathrm{Cov}(X,Y)$$

$$\mathrm{Cov}(X+Z,Y) = \mathrm{Cov}(X,Y) + \mathrm{Cov}(Z,Y)$$

$$\mathrm{Cov}(X,X) = \mathrm{Var}(X)$$

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathbb{E}\big[(X+Y)^2\big] - \mathbb{E}\big[(X+Y)\big]^2 \\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\mathbb{E}[Y] \\ &\qquad + 2\,\mathrm{Cov}(X,Y)\end{aligned}$$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$$|\mathrm{Corr}(X,Y)| \le 1$$

$|\mathrm{Corr}(X,Y)| = 1$ if and only if $Y = a + bX$, $a,b \in \mathbb{R}$

Independence

$X \perp Y$ iff $\{X \leq x\} \underset{}{\overset{\text{mutually}}{\perp}} \{Y \leq y\}$ $\forall x, y \in \mathbb{R}$

$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, $A \perp B$

$$F_{XY}(x,y) = F_X(x)\, F_Y(y)$$

Equivalently: $f_{XY}(x,y) = f_X(x) f_Y(y)$

Example: $X \perp Y$ and cts

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} y f_Y(y) \left(\int_{-\infty}^{\infty} x f_X(x)\, dx \right) dy$$

$$= \mathbb{E}[X] \int_{-\infty}^{\infty} y f_Y(y)\, dy = \mathbb{E}[X]\,\mathbb{E}[Y]$$

$\Rightarrow$ if $X \perp Y$ $\mathrm{Cov}(X,Y) = 0$

Suppose $g(X,Y) = h(X)k(Y)$

$X \perp Y$: $\mathbb{E}[g(X,Y)] = \mathbb{E}[h(X)]\,\mathbb{E}[k(Y)]$

$$\mathbb{E}\left[e^{tX+uY}\right] = \mathbb{E}\left[e^{tX}\right]\mathbb{E}\left[e^{uY}\right] = M_X(t)M_Y(u)$$

Sums of RVs

$Z = X + Y$

$$f_Z(z) = \begin{cases} \sum_u f_{XY}(u, z-u) & \text{discrete} \\ \int_{-\infty}^{\infty} f_{XY}(u, z-u)\,du & \text{cts} \end{cases}$$

Proof (Discrete)

$\{Z = z\}$ $\quad Z = X + Y$

$$\{Z = z\} = \bigcup_u \{X = u, Y = z-u\}$$

$$\mathbb{P}(Z=z) = \sum_u \mathbb{P}(X=u, Y=z-u) = \sum_u f_{XY}(u, z-u)$$

If $X \perp Y$, $Z = X + Y$

$$f_Z(z) = \begin{cases} \sum_u f_X(u) f_Y(z-u) \\ \int_{-\infty}^{\infty} f_X(u) f_Y(z-u)\,du \end{cases}$$

"Convolution of X and Y"

$$f_Z = f_X * f_Y$$

$$\underset{n\times 1}{X} : \Omega \to \mathbb{R}^n$$

$$\mathrm{Var}(\underset{n\times 1}{X}) = \mathbb{E}\Big[\underbrace{(\underset{n\times 1}{X} - \underset{n\times 1}{\mathbb{E}[X]})(X - \mathbb{E}[X])^T}_{n\times n}\Big] =$$

$$= \begin{pmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2), & \cdots & \mathrm{Cov}(X_1, X_n) \\ \mathrm{Cov}(X_2, X_1) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathrm{Cov}(X_n, X_1) & \cdots & \cdots & \mathrm{Var}(X_n) \end{pmatrix}$$

If X_i's are independent then $\mathrm{Var}(X)$ is diagonal

identically distributed with $\mathrm{Var}(X_i) = \sigma^2$, $\mathrm{Var}(X) = \sigma^2 I_{n\times n}$

$\underset{m\times 1}{\mathrm{Var}(X)}$ is PSD $\Leftrightarrow$ $\forall a \in \mathbb{R}^m$ $a^T \mathrm{Var}(X) a \geq 0$,

Proof: $0 \leq \mathrm{Var}(\underset{1\times m}{a^T} \underset{m\times 1}{X}) = \mathbb{E}\left[(a^T X - \mathbb{E}[a^T X])(a^T X - \mathbb{E}[a^T X])^T\right]$ $\Big|$ $(a^T X)^T = X^T a$

$$= \mathbb{E}\left[a^T (X - \mathbb{E}[X])(X - \mathbb{E}[X])^T a\right]$$

$$= a^T \mathrm{Var}(X) a$$

Conditional Distribution

$\mathbb{P}(B) > 0$, $\quad \mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

The PDF/PMF of Y given $X = x$ is given

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Is this valid?

$$\sum_y f_{Y|X}(y|x) = \sum_y \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1}{f_X(x)} \sum_y f_{XY}(x,y)$$

$$= 1$$

$$F_{Y|X}(y|x) = \begin{cases} \sum_{y' \leq y} f_{Y|X}(y'|x) \\ \int_{-\infty}^{y} f_{Y|X}(y'|x)\, dy' \end{cases}$$

Hurricanes

Y # hurricanes reaching land

N # hurricanes formed in ocean

$Y|N=n \sim \text{Bin}(n,p) \qquad \mathbb{E}[Y|N=n] = np$

$N \sim \text{Poisson}(\lambda) \qquad \mathbb{E}[N] = \lambda$

One can show $Y \sim \text{Poisson}(\lambda p)$

Conditional Expectation

The conditional expectation of Y given $X=x$

$$\mathbb{E}[Y|X=x] = \begin{cases} \sum_y y f_{Y|X}(y|x) & \text{discrete} \\ \int_{-\infty}^{\infty} y f_{Y|X}(y|x) & \text{cts} \end{cases}$$

$\psi(x) = \mathbb{E}[Y|X=x]$

The conditional expectation of Y given X is $\psi(X)$ and we write

$$\mathbb{E}[Y|X]$$

Law of Iterated Expectations :

$$\mathbb{E}\left[\mathbb{E}[Y|X]\right] = \mathbb{E}[Y]$$

Proof :

$$\int_{-\infty}^{\infty} \mathbb{E}[Y|X=x]\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy\, f_X(x)\, dx \qquad \Big| \quad f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \mathbb{E}[Y]$$

$$g(X,Y) = h(X)k(Y)$$

$$\mathbb{E}\left[g(X,Y)\,|\,X=x\right] = h(x)\,\mathbb{E}\left[k(Y)\,|\,X=x\right]$$

$$\mathbb{E}\left[g(X,Y)\,|\,X\right] = h(X)\,\mathbb{E}\left[k(Y)\,|\,X\right]$$